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ON CURVILINEAR COORDINATES.

BY BENJAMIN WILLIAMSON, M.A., F.T.C.D., F.R.S., a Vice-President of the Academy.

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XV.

ON CURVILINEAR COORDINATES.

By BENJAMIN WILLIAMSON, M.A., F.T.C.D., F.R.S., a Vice-President of the Academy.

[Read June 23, 1890.]

Section 1.—*Application to Surfaces.*

1. Preliminary.—The general theory of Curvilinear Coordinates is, I believe, due to Lamé, who first discussed their properties in the *Comptes rendus*, 1838, and in Liouville's *Journal des Mathématiques*, 1840. His results were afterwards produced in a comprehensive form in his well-known work entitled *Leçons sur les Coordonnées curvilignes*, Paris, 1859.

In this Paper I propose, by a somewhat different method, to arrive at the principal results established by Lamé, and also to treat of curves traced on a fixed surface, thus establishing, in a new form, Gauss's well-known expression for the *measure of curvature* at any point on a surface, along with some kindred theorems.

2. Curvilinear Coordinates.—Suppose that three systems of surfaces are represented by the equations

$$f_1(x, y, z) = a_1, \quad f_2(x, y, z) = a_2, \quad f_3(x, y, z) = a_3, \qquad (1)$$

where a_1, a_2, a_3 are variable parameters; then through any point P there can be drawn one, and but one, surface for each system, and the particular values of a_1, a_2, a_3 at the point may be called the *curvilinear coordinates* of the point relative to the system of surfaces.

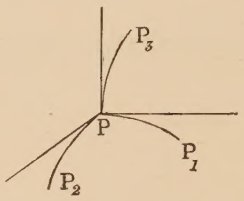
Let us denote by S_1, S_2, S_3 the three surfaces which pass through P; also by s_1 the curve of intersection of the surfaces S_2 and S_3, with similar definitions of s_2 and s_3. Then ds_1, ds_2, ds_3 represent the elements of the three curves which pass through P.

Again, since a_2 and a_3 are both constant along the curve s_1, &c., we see that in all cases we must have

$$ds_1 = k_1 da_1, \quad ds_2 = k_2 da_2, \quad ds_3 = k_3 da_3, \tag{2}$$

where k_1, k_2, k_3 are functions of the curvilinear coordinates of the point.

Next let PP_1, PP_2, PP_3 represent the elements ds_1, ds_2, ds_3, respectively; also, let l_1, m_1, n_1, be the direction cosines of the tangent to ds_1 at P, and let l_2, m_2, n_2; l_3, m_3, n_3 be the corresponding cosines for the tangents to PP_2 and PP_3, respectively. Then we have



$$\frac{dx}{ds_1} = l_1, \quad \frac{dy}{ds_1} = m_1, \quad \frac{dz}{ds_1} = n_1, \text{ &c.} \tag{3}$$

Also, regarding x, y, z as functions of a_1, a_2, a_3, we have

$$\frac{dx}{da_1} = \frac{dx}{ds_1}\frac{ds_1}{da} = l_1 k_1, \quad \frac{dy}{da_1} = m_1 k_1, \quad \frac{dz}{da_1} = n_1 k_1, \tag{4}$$

and also

$$\frac{dx}{da_2} = l_2 k_2, \quad \frac{dy}{da_2} = m_2 k_2, \quad \frac{dz}{da_2} = n_2 k_2, \text{ &c.}$$

Again, since

$$\frac{d}{da_2}\left(\frac{dx}{da_1}\right) = \frac{d}{da_1}\left(\frac{dx}{da_2}\right),$$

we get

$$\frac{d}{da_2}(l_1 k_1) = \frac{d}{da_1}(l_2 k_2),$$

consequently we have the following system of equations:—

$$\left.\begin{aligned} k_2 \frac{d}{ds_2}(l_1 k_1) &= k_1 \frac{d}{ds_1}(l_2 k_2) \\ k_2 \frac{d}{ds_2}(m_1 k_1) &= k_1 \frac{d}{ds_1}(m_2 k_2) \\ k_2 \frac{d}{ds_2}(n_1 k_1) &= k_1 \frac{d}{ds_1}(n_2 k_2) \end{aligned}\right\}. \tag{5}$$

Again, if V be any function, we have

$$\left.\begin{aligned} \frac{dV}{ds_1} &= l_1\frac{dV}{dx} + m_1\frac{dV}{dy} + n_1\frac{dV}{dz} \\ \frac{dV}{ds_2} &= l_2\frac{dV}{dx} + m_2\frac{dV}{dy} + n_2\frac{dV}{dz} \\ \frac{dV}{ds_3} &= l_3\frac{dV}{dx} + m_3\frac{dV}{dy} + n_3\frac{dV}{dz} \end{aligned}\right\}. \qquad (6)$$

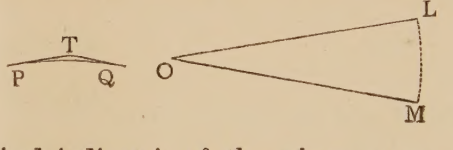
Also

$$\left.\begin{aligned} \frac{dV}{dx} &= l_1\frac{dV}{ds_1} + l_2\frac{dV}{ds_2} + l_3\frac{dV}{ds_3} \\ \frac{dV}{dy} &= m_1\frac{dV}{ds_1} + m_2\frac{dV}{ds_2} + m_3\frac{dV}{ds_3} \\ \frac{dV}{dz} &= n_1\frac{dV}{ds_1} + n_2\frac{dV}{ds_2} + n_3\frac{dV}{ds_3} \end{aligned}\right\}. \qquad (7)$$

3. Any function of x, y, z may be regarded as a function of the curvilinear coordinates $\alpha_1, \alpha_2, \alpha_3$; but it should be observed that we cannot regard it as a function of s_1, s_2, s_3. Accordingly, while $\alpha_1, \alpha_2, \alpha_3$ may be taken as a system of independent variables, we cannot assume s_1, s_2, s_3 as such a system.

For instance, we have in all cases $\frac{d}{d\alpha_1}\left(\frac{dV}{d\alpha_2}\right) = \frac{d}{d\alpha_2}\left(\frac{dV}{d\alpha_1}\right)$, but the expressions $\frac{d}{ds_1}\left(\frac{dV}{ds_2}\right)$ and $\frac{d}{ds_2}\left(\frac{dV}{ds_1}\right)$ are not equivalent.

4. **On Curves of Double Curvature.**—If from the origin O a series of lines be drawn parallel to the tangents to any curve of double curvature, these lines will intersect a unit sphere, described round O as centre, in a curve, which has been called by Serret the *spherical indicatrix* of the given curve.



Thus, let l, m, n be the direction cosines of the tangent at P to the curve, and let L be the corresponding point on the sphere; then will l, m, n be the coordinates of the point L, relative to our rectangular system of axes.

Again, the coordinates of a consecutive point M on the spherical indicatrix are

$$l+\frac{dl}{ds}ds, \quad m+\frac{dm}{ds}ds, \quad n+\frac{dn}{ds}ds;$$

hence we have

$$LM^2 = ds^2\left\{\left(\frac{dl}{ds}\right)^2+\left(\frac{dm}{ds}\right)^2+\left(\frac{dn}{ds}\right)^2\right\}.$$

But LM is the circular measure of the angle of contingence, ϵ, for the curve PQ; accordingly, if ρ be the radius of absolute curvature at P, we have

$$\frac{1}{\rho^2}=\left(\frac{dl}{ds}\right)^2+\left(\frac{dm}{ds}\right)^2+\left(\frac{dn}{ds}\right)^2. \tag{8}$$

Again, in the limit LM is parallel to the radius of curvature at P, and hence we have

$$\lambda:\mu:\nu=\frac{dl}{ds}:\frac{dm}{ds}:\frac{dn}{ds}, \tag{9}$$

where λ, μ, ν are the direction cosines of the *principal normal* to the curve.

Consequently, by (8), we have

$$\frac{\lambda}{\rho}=\frac{dl}{ds}, \quad \frac{\mu}{\rho}=\frac{dm}{ds}, \quad \frac{\nu}{\rho}=\frac{dn}{ds}. \tag{10}$$

5. **Orthogonal Surfaces.**—The most important case of curvilinear coordinates is that discussed by Lamé, for which the surfaces S_1, S_2, S_3 are mutually orthogonal.

In that case we have the following system of equations:—

$$\left.\begin{aligned} l_1l_2+m_1m_2+n_1n_2&=0\\ l_3l_1+m_3m_1+n_3n_1&=0\\ l_2l_3+m_2m_3+n_2n_3&=0 \end{aligned}\right\}. \tag{11}$$

And also

$$l_1m_2-m_1l_2=n_3, \quad n_1l_2-l_1n_2=m_3, \quad m_1n_2-n_1m_2=l_3, \tag{12}$$

with similar equations got by interchange of suffixes.

Equations (11) may also be written in the form

$$\left.\begin{aligned} \frac{df_1}{dx}\frac{df_2}{dx}+\frac{df_1}{dy}\frac{df_2}{dy}+\frac{df_1}{dz}\frac{df_2}{dz}&=0\\ \frac{df_3}{dx}\frac{df_1}{dx}+\frac{df_3}{dy}\frac{df_1}{dy}+\frac{df_3}{dz}\frac{df_1}{dz}&=0\\ \frac{df_2}{dx}\frac{df_3}{dx}+\frac{df_2}{dy}\frac{df_3}{dy}+\frac{df_2}{dz}\frac{df_3}{dz}&=0 \end{aligned}\right\}. \qquad (13)$$

6. **Dupin's Theorem.**—I shall now proceed to give a general demonstration of Dupin's well-known theorem that *when three systems of surfaces cut orthogonally their curves of intersection are lines of curvature on the surfaces.*

I commence by finding the condition that ds_3 should be an element of a line of curvature on the surface S_1.

By hypothesis l_1, m_1, n_1 are the direction cosines of the normal to the surface S_1 at P. Hence it follows that, neglecting indefinitely small quantities of the second order,

$$l_1+\frac{dl_1}{ds_3}ds_3, \quad m_1+\frac{dm_1}{ds_3}ds_3, \quad n_1+\frac{dn_1}{ds_3}ds_3,$$

are the direction cosines of the normal to S_1 at the point P_3.

Now, if PP_3 be an element of a line of curvature, the normals at P and P_3 must intersect, *i.e.* the normal at P_3 must be perpendicular to the line $l_2m_2n_2$.

Accordingly, the required condition is

$$l_2\left(l_1+\frac{dl_1}{ds_3}ds_3\right)+m_2\left(m_1+\frac{dm_1}{ds_3}ds_3\right)+n_2\left(n_1+\frac{dn_1}{ds_3}ds_3\right)=0,$$

or

$$l_2\frac{dl_1}{ds_3}+m_2\frac{dm_1}{ds_3}+n_2\frac{dn_1}{ds_3}=0. \qquad (14)$$

I now proceed to give a general proof of this latter equation, and thus establish Dupin's theorem.

In the first place we have

$$\frac{df_1}{dx}=\frac{da_1}{dx}=\frac{da_1}{ds_1}\frac{ds_1}{dx}=\frac{l_1}{k_1}, \quad \frac{df_1}{dy}=\frac{m_1}{k_1}, \quad \frac{df_1}{dz}=\frac{n_1}{k_1}, \text{ \&c.} \qquad (15)$$

Hence it follows that the expression

$$l_1 dx + m_1 dy + n_1 dz$$

is made integrable by the factor k_1; consequently we must have

$$\left(l_1 \frac{d}{dy} - m_1 \frac{d}{dx}\right) n_1 + \left(n_1 \frac{d}{dx} - l_1 \frac{d}{dz}\right) m_1 + \left(m_1 \frac{d}{dz} - n_1 \frac{d}{dy}\right) l_1 = 0. \quad (16)$$

Transforming by (7) this becomes

$$l_2 \frac{dl_1}{ds_3} + m_2 \frac{dm_1}{ds_3} + n_2 \frac{dn_1}{ds_3} = l_3 \frac{dl_1}{ds_2} + m_3 \frac{dm_1}{ds_2} + n_3 \frac{dn_1}{ds_2}. \quad (17)$$

Again, if h denote a small increment of x, the surfaces

$$f_2(x+h, y, z) = \text{const.}, \quad f_3(x+h, y, z) = \text{const.},$$

intersect orthogonally.

These surfaces may be written

$$f_2 + h \frac{df_2}{dx} = \text{const.}, \quad f_3 + h \frac{df_3}{dx} = \text{const.},$$

or

$$f_2 + h \frac{l_2}{k_2} = \text{const.}, \quad f_3 + h \frac{l_3}{k_3} = \text{const.}$$

Consequently by (13) we have

$$\frac{df_3}{dx} \frac{d}{dx}\left(\frac{l_2}{k_2}\right) + \frac{df_3}{dy} \frac{d}{dy}\left(\frac{l_2}{k_2}\right) + \frac{df_3}{dz} \frac{d}{dz}\left(\frac{l_2}{k_2}\right) + \frac{df_2}{dx} \frac{d}{dx}\left(\frac{l_3}{k_3}\right) + \frac{df_2}{dy} \frac{d}{dy}\left(\frac{l_3}{k_3}\right)$$

$$+ \frac{df_2}{dz} \frac{d}{dz}\left(\frac{l_3}{k_3}\right) = 0.$$

This, by (15), readily becomes

$$\frac{1}{k_3} \frac{d}{ds_3}\left(\frac{l_2}{k_2}\right) + \frac{1}{k_2} \frac{d}{ds_2}\left(\frac{l_3}{k_3}\right) = 0,$$

or

$$k_2 \frac{d}{ds_3}\left(\frac{l_2}{k_2}\right) + k_3 \frac{d}{ds_2}\left(\frac{l_3}{k_3}\right) = 0.$$

In like manner we get

$$k_2 \frac{d}{ds_3}\left(\frac{m_2}{k_2}\right) + k_3 \frac{d}{ds_2}\left(\frac{m_3}{k_3}\right) = 0, \quad k_2 \frac{d}{ds_3}\left(\frac{n_2}{k_2}\right) + k_3 \frac{d}{ds_2}\left(\frac{n_3}{k_3}\right) = 0. \qquad (18)$$

If we multiply the three last equations by l_1, m_1, n_1, respectively, and add, we readily get

$$l_1 \frac{dl_2}{ds_3} + m_1 \frac{dm_2}{ds_3} + n_1 \frac{dn_2}{ds_3} + l_1 \frac{dl_3}{ds_2} + m_1 \frac{dm_3}{ds_2} + n_1 \frac{dn_3}{ds_2} = 0,$$

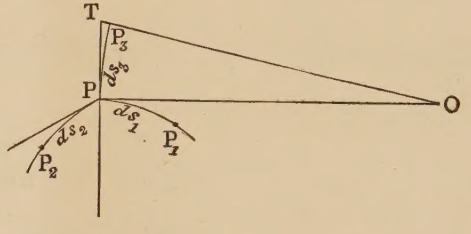
or,
$$l_2 \frac{dl_1}{ds_3} + m_2 \frac{dm_1}{ds_3} + n_2 \frac{dn_1}{ds_3} + l_3 \frac{dl_1}{ds_2} + m_3 \frac{dm_1}{ds_2} + n_3 \frac{dn_1}{ds_2} = 0.$$

Hence by (17), we see that

$$l_2 \frac{dl_1}{ds_3} + m_2 \frac{dm_1}{ds_3} + n_2 \frac{dn_1}{ds_3} = 0.$$

This establishes equation (14), and accordingly proves the theorem (Dupin, *Développements de Géométrie*, 1813, p. 330).

7. **Expressions for Radii of Curvature.**—We have seen that ds_1 is an element of a line of curvature on each of the surfaces S_2 and S_3; hence also ds_2 is an element of a line of curvature on S_1 and S_3, &c.



Consequently the normal at P_3 to the surface S_1 intersects PO, the tangent to ds_1. Again, the angle between P_3O and the tangent PT is equal to $\frac{\pi}{2} + POP_3$; hence, by (8),

$$l_3\left(l_1 + \frac{dl_1}{ds_3}\,ds_3\right) + m_3\left(m_1 + \frac{dm_1}{ds_3}\,ds_3\right) + n_3\left(n_1 + \frac{dn_1}{ds_3}\,ds_3\right) = -\sin POP_3 = -\frac{ds_3}{OP};$$

therefore
$$l_3 \frac{dl_1}{ds_3} + m_3 \frac{dm_1}{ds_3} + n_3 \frac{dn_1}{ds_3} = -\frac{1}{OP},$$

or
$$l_1 \frac{dl_3}{ds_3} + m_1 \frac{dm_3}{ds_3} + n_1 \frac{dn_3}{ds_3} = \frac{1}{OP}.$$

Let us now denote the principal radii of curvature of the surface S_1 at P by ${}_2R_1$ and ${}_3R_1$, where ${}_2R_1$ corresponds to the element ds_2, and ${}_3R_1$ to ds_3; then the last equation may be written

$$l_1 \frac{dl_3}{ds_3} + m_1 \frac{dm_3}{ds_3} + n_1 \frac{dn_3}{ds_3} = \frac{1}{{}_3R_1}, \tag{19}$$

and also

$$l_3 \frac{dl_1}{ds_3} + m_3 \frac{dm_1}{ds_3} + n_3 \frac{dn_1}{ds_3} = -\frac{1}{{}_3R_1}, \tag{20}$$

with corresponding expressions for the other radii of curvature.

Next, let equations (5) be respectively multiplied by l_1, m_1, and n_1, then we get, by addition,

$$\frac{dk_1}{ds_2} = k_1 \left(l_1 \frac{dl_2}{ds_1} + m_1 \frac{dm_1}{ds_1} + n_1 \frac{dn_2}{ds_1} \right);$$

hence by (19),

$$\frac{1}{{}_1R_2} = l_2 \frac{dl_1}{ds_1} + m_2 \frac{dm_1}{ds_1} + n_2 \frac{dn_1}{ds_1} = -\frac{1}{k_1} \frac{dk_1}{ds_2}, \tag{21}$$

together with similar equations by interchange of suffixes.

8. Again, since PP_2 is perpendicular to P_3O, we have immediately by (12),

$$l_3 \frac{dm_1}{ds_3} - m_3 \frac{dl_1}{ds_3} = 0, \text{ \&c.}$$

Consequently

$$l_3 : m_3 : n_3 = \frac{dl_1}{ds_3} : \frac{dm_1}{ds_3} : \frac{dn_1}{ds_3}. \tag{22}$$

Hence, by (20), we have

$$\frac{1}{l_3} \frac{dl_1}{ds_3} = \frac{1}{m_3} \frac{dm_1}{ds_3} = \frac{1}{n_3} \frac{dn_1}{ds_3} = -\frac{1}{{}_3R_1}. \tag{23}$$

Also

$$\left(\frac{dl_1}{ds_3}\right)^2 + \left(\frac{dm_1}{ds_3}\right)^2 + \left(\frac{dn_1}{ds_3}\right)^2 = \left(\frac{1}{{}_3R_1}\right)^2.$$ Compare § 4.

The following results follow immediately:—

$$\frac{dl_1}{ds_1} \frac{dl_3}{ds_1} + \frac{dm_1}{ds_1} \frac{dm_3}{ds_1} + \frac{dn_1}{ds_1} \frac{dn_3}{ds_1} = 0,$$

$$\frac{dl_1}{ds_1} \frac{dl_2}{ds_2} + \frac{dm_1}{ds_1} \frac{dm_2}{ds_2} + \frac{dn_1}{ds_1} \frac{dn_2}{ds_2} = \frac{1}{{}_1R_3 \, {}_2R_3},$$

$$\left(\frac{dl_1}{ds_1}\right)^2 + \left(\frac{dm_1}{ds_1}\right)^2 + \left(\frac{dn_1}{ds_1}\right)^2 = \frac{1}{{}_1R_3{}^2} + \frac{1}{{}_1R_2{}^2},$$

together with similar equations by interchange of suffixes.

9. **Transformation of** $\dfrac{d^2V}{dx^2}+\dfrac{d^2V}{dy^2}+\dfrac{d^2V}{dz^2}$.

Let us next consider the transformation to curvilinear coordinates of the well-known Laplacian expression

$$\nabla^2 V \equiv \frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}.$$

Here, since $$\frac{dV}{dx}=l_1\frac{dV}{ds_1}+l_2\frac{dV}{ds_2}+l_3\frac{dV}{ds_3},$$

we have $$\frac{d^2V}{dx^2}=\left(l_1\frac{d}{ds_1}+l_2\frac{d}{ds_2}+l_3\frac{d}{ds_3}\right)\left(l_1\frac{dV}{ds_1}+l_2\frac{dV}{ds_2}+l_3\frac{dV}{ds_3}\right).$$

Hence
$$\nabla^2 V=\left(l_1\frac{d}{ds_1}+l_2\frac{d}{ds_2}+l_3\frac{d}{ds_3}\right)\left(l_1\frac{dV}{ds_1}+l_2\frac{dV}{ds_2}+l_3\frac{dV}{ds_3}\right)$$
$$+\left(m_1\frac{d}{ds_1}+m_2\frac{d}{ds_2}+m_3\frac{d}{ds_3}\right)\left(m_1\frac{dV}{ds_1}+m_2\frac{dV}{ds_2}+m_3\frac{dV}{ds_3}\right)$$
$$+\left(n_1\frac{d}{ds_1}+n_2\frac{d}{ds_2}+n_3\frac{d}{ds_3}\right)\left(n_1\frac{dV}{ds_1}+n_2\frac{dV}{ds_2}+n_3\frac{dV}{ds_3}\right).$$

Consequently, by (21), we get

$$\nabla^2 V=\frac{d}{ds_1}\left(\frac{dV}{ds_1}\right)+\frac{d}{ds_2}\left(\frac{dV}{ds_2}\right)+\frac{d}{ds_3}\left(\frac{dV}{ds_3}\right)+\frac{dV}{ds_1}\left(\frac{1}{k_2}\frac{dk_2}{ds_1}+\frac{1}{k_3}\frac{dk_3}{ds_1}\right)$$
$$+\frac{dV}{ds_2}\left(\frac{1}{k_1}\frac{dk_1}{ds_2}+\frac{1}{k_3}\frac{dk_3}{ds_2}\right)+\frac{dV}{ds_3}\left(\frac{1}{k_1}\frac{dk_1}{ds_3}+\frac{1}{k_2}\frac{dk_2}{ds_3}\right).$$

Again, $$\frac{d}{ds_1}\left(\frac{dV}{ds_1}\right)+\frac{dV}{ds_1}\left(\frac{1}{k_2}\frac{dk_2}{ds_1}+\frac{1}{k_3}\frac{dk_3}{ds_1}\right)=\frac{1}{k_2k_3}\frac{d}{ds_1}\left(k_2k_3\frac{dV}{ds_1}\right);$$

hence $$\nabla^2 V=\frac{1}{k_2k_3}\frac{d}{ds_1}\left(k_2k_3\frac{dV}{ds_1}\right)+\frac{1}{k_3k_1}\frac{d}{ds_2}\left(k_3k_1\frac{dV}{ds_2}\right)+\frac{1}{k_1k_2}\frac{d}{ds_3}\left(k_1k_2\frac{dV}{ds_3}\right).$$

This may also be written

$$\nabla^2 V=\frac{1}{k_1k_2k_3}\left[\frac{d}{d\alpha_1}\left(\frac{k_2k_3}{k_1}\frac{dV}{d\alpha_1}\right)+\frac{d}{d\alpha_2}\left(\frac{k_3k_1}{k_2}\frac{dV}{d\alpha_2}\right)+\frac{d}{d\alpha_3}\left(\frac{k_1k_2}{k_3}\frac{dV}{d\alpha_3}\right)\right] \quad (24)$$

This is Lamé's well-known transformation.

In polar coordinates we have

$$ds_1 = dr, \quad ds_2 = r\,d\theta, \quad ds_3 = r\sin\theta\,d\phi, \tag{25}$$

and consequently

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = \nabla^2 V$$

$$= \frac{1}{r^2}\left\{\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) + \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dV}{d\theta}\right) + \frac{1}{\sin^2\theta}\frac{d^2V}{d\phi^2}\right\}. \tag{26}$$

Again, in cylindrical coordinates, we have

$$\nabla^2 V = \frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right) + \frac{1}{r^2}\frac{d^2V}{d\theta^2} + \frac{d^2V}{dz^2}. \tag{27}$$

SECTION 2.—*Curves traced on a Fixed Surface.*

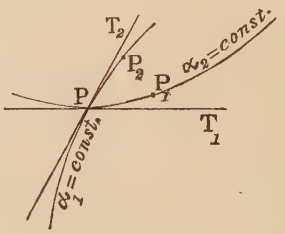
10. In the general case if one of the surfaces, S_3 suppose, be regarded as fixed, the coordinates of all points on the surface may be regarded as functions of two independent variables, α_1 and α_2. In this case the equations

$$\alpha_1 = \text{const.}, \qquad \alpha_2 = \text{const.},$$

may be treated, according to Gauss' method, as representing two systems of curves traced on the surface.

Let ds_1 and ds_2 represent the elements of the curves passing through any point P on the surface, then we shall have

$$ds_1 = k_1 d\alpha_1, \quad ds_2 = k_2 d\alpha_2. \tag{1}$$



Again, as before, let l_1, m_1, n_1 be the direction cosines of the tangent PT_1, and l_2, m_2, n_2 the corresponding cosines for PT_2; also let l_3, m_3, n_3 be the direction cosines of the normal at P to the surface, then we have

$$\left.\begin{aligned} l_1l_3 + m_1m_3 + n_1n_3 &= 0 \\ l_2l_3 + m_2m_3 + n_2n_3 &= 0 \\ l_1l_2 + m_1m_2 + n_1n_2 &= \cos\omega \end{aligned}\right\}, \tag{2}$$

where ω is the angle between the tangents PT_1 and PT_2.

We also have

$$l_1 m_2 - m_1 l_2 = n_3 \sin\omega, \quad n_1 l_2 - l_1 n_2 = m_3 \sin\omega, \quad m_1 n_2 - n_1 m_2 = l_3 \sin\omega. \quad (3)$$

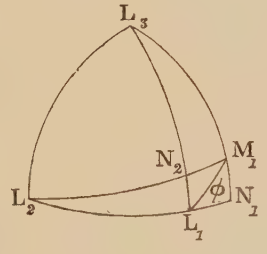
Again, as in Art. 2, we have

$$\frac{dx}{ds_1} = l_1, \quad \frac{dy}{ds_1} = m_1, \quad \frac{dz}{ds_1} = n_1, \text{ \&c.},$$

and also

$$\left.\begin{aligned} \frac{dx}{da_1} &= k_1 l_1, \quad \frac{dy}{da_1} = k_1 m_1, \quad \frac{dz}{da_1} = k_1 n_1, \\ \frac{dx}{da_2} &= k_2 l_2, \quad \frac{dy}{da_2} = k_2 m_2, \quad \frac{dz}{da_2} = k_2 n_2 \end{aligned}\right\}. \quad (4)$$

11. **Spherical Representation.**—If, as in Art. 4, we suppose the directions of the tangents PT_1, PT_2, and of the normal, to be represented respectively by the points L_1, L_2, L_3 on the unit sphere; and if M_1 represents the direction of the consecutive tangent to ds_1; then $M_1L_1N_1$ is the angle between the osculating plane to the curve and the tangent plane to the surface; let $\phi = M_1L_1N_1$, $\omega = L_2L_1$, $L_1M_1 = \dfrac{ds_1}{\rho}$, where ρ is the radius of *absolute curvature* of the arc ds_1; then we have



$$l_3 \frac{dl_1}{ds_1} + m_3 \frac{dm_1}{ds_1} + n_3 \frac{dn_1}{ds_1} = \frac{\cos L_3 M_1}{ds_1} = \frac{M_1 N_1}{ds_1} = \frac{\sin\phi}{\rho} = \frac{1}{\rho_1}, \quad (5)$$

where ρ_1 is the radius of curvature of the normal section through ds_1.

Also $$\cos L_2 M_1 = l_2\left(l_1 + \frac{dl_1}{ds_1}\,ds_1\right) + m_2\left(m_1 + \frac{dm_1}{ds_1}\,ds_1\right) + n_2\left(n_1 + \frac{dn_1}{ds_1}\,ds_1\right)$$

$$= \cos\omega + ds_1\left(l_2\frac{dl_1}{ds_1} + m_2\frac{dm_1}{ds_1} + n_2\frac{dn_1}{ds_1}\right).$$

But, by Spherical Trigonometry,

$$\cos L_2 M_1 = \cos\omega \cos L_1 M_1 - \sin\omega \sin L_1 M_1 \cos\phi = \cos\omega - \sin\omega \cos\phi \frac{ds_1}{\rho};$$

hence we get

$$l_2 \frac{dl_1}{ds_1} + m_2 \frac{dm_1}{ds_1} + n_2 \frac{dn_1}{ds_1} = -\sin\omega \frac{\cos\phi}{\rho}. \quad (6)$$

Again, $\frac{\cos\phi}{\rho}$ plainly measures the geodesic curvature of the arc ds_1 (Salmon, *Geometry of Three Dimensions*, p. 360); hence, if this radius of geodesic curvature be represented by r_1, we get

$$l_2\frac{dl_1}{ds_1}+m_2\frac{dm_1}{ds_1}+n_2\frac{dn_1}{ds_1}=-\frac{\sin\omega}{r_1}. \tag{7}$$

In like manner

$$l_1\frac{dl_2}{ds_2}+m_1\frac{dm_2}{ds_2}+n_1\frac{dn_2}{ds_2}=-\frac{\sin\omega}{r_2}, \tag{8}$$

where $\frac{1}{r_2}$ is the geodesic curvature of the arc ds_2.

Again, equation (5) may be written in the determinant form

$$\frac{1}{\rho_1}=\frac{1}{\sin\omega}\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ \frac{dl_1}{ds_1}, & \frac{dm_1}{ds_1}, & \frac{dn_1}{ds_1} \end{vmatrix}. \tag{9}$$

I will subsequently exemplify the importance of this latter form.

12. **Expression for Geodesic Curvature.**—Next, from equations (4), we get, as in § 2,

$$\left.\begin{aligned} k_2\frac{d}{ds_2}(k_1l_1)&=k_1\frac{d}{ds_1}(k_2l_2)\\ k_2\frac{d}{ds_2}(k_1m_1)&=k_1\frac{d}{ds_1}(k_2m_2)\\ k_2\frac{d}{ds_2}(k_1n_1)&=k_1\frac{d}{ds_1}(k_2n_2) \end{aligned}\right\}. \tag{10}$$

If we multiply these respectively by l_3, m_3, n_3, and add, we obtain

$$l_3\frac{dl_1}{ds_2}+m_3\frac{dm_1}{ds_2}+n_3\frac{dn_1}{ds_2}=l_3\frac{dl_2}{ds_1}+m_3\frac{dm_2}{ds_1}+n_3\frac{dn_2}{ds_1}. \tag{11}$$

Again, if we multiply by l_1, m_1, n_1, respectively, we get, in like manner,

$$\frac{1}{k_1}\frac{dk_1}{ds_2}=l_1\frac{dl_2}{ds_1}+m_1\frac{dm_2}{ds_1}+n_1\frac{dn_2}{ds_1}+\frac{\cos\omega}{k_2}\frac{dk_2}{ds_1}. \tag{12}$$

Also, since

$$l_2\frac{dl_1}{ds_1}+m_2\frac{dm_1}{ds_1}+n_2\frac{dn_1}{ds_1}+l_1\frac{dl_2}{ds_1}+m_1\frac{dm_2}{ds_1}+n_1\frac{dn_2}{ds_1}=-\sin\omega\frac{d\omega}{ds_1}, \qquad (13)$$

the preceding equation gives

$$l_2\frac{dl_1}{ds_1}+m_2\frac{dm_1}{ds_1}+n_2\frac{dn_1}{ds_1}=-\frac{1}{k_1}\frac{dk_1}{ds_2}+\frac{1}{k_2}\frac{d(k_2\cos\omega)}{ds_1}.$$

Hence

$$\frac{\sin\omega}{r_1}=\frac{1}{k_1}\frac{dk_1}{ds_2}-\frac{1}{k_2}\frac{d(k_2\cos\omega)}{ds_1}. \qquad (14)$$

When the curves intersect orthogonally this reduces to

$$\frac{1}{r_1}=\frac{1}{k_1}\frac{dk_1}{ds_2}. \qquad (15)$$

In like manner

$$\frac{\sin\omega}{r_2}=\frac{1}{k_2}\frac{dk_2}{ds_1}-\frac{1}{k_1}\frac{d(k_1\cos\omega)}{ds_2}.$$

Again, if α_2 be a geodesic we have $\phi=\frac{\pi}{2}$, and consequently along a geodesic we have the equation

$$l_2\frac{dl_1}{ds_1}+m_2\frac{dm_1}{ds_1}+n_2\frac{dn_1}{ds_1}=0. \qquad (16)$$

This latter equation follows also immediately from the consideration that for a geodesic the line whose direction cosines are in the proportion of $\frac{dl_1}{ds_1}:\frac{dm_1}{ds_1}:\frac{dn_1}{ds_1}$ is perpendicular to the tangent plane to the surface.

13. **Lemma.**—In general, if the lines lmn, $l'm'n'$, $l''m''n''$ be parallel to a plane, we have

$$\left.\begin{aligned}\lambda l+\mu m+\nu n&=0,\\ \lambda l'+\mu m'+\nu n'&=0,\\ \lambda l''+\mu m''+\nu n''&=0,\end{aligned}\right\}$$

where λ, μ, ν are the direction cosines of a normal to the plane.

Hence, for three coplanar directions we have the determinant relation

$$\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ l'', & m'', & n'' \end{vmatrix} = 0. \qquad (17)$$

14. **Line of Curvature.**—If s_1 be a line of curvature on the surface, the lines l_1, m_1, n_1; l_3, m_3, n_3; and $l_3 + \frac{dl_3}{ds_1} ds_1$, $m_3 + \frac{dm_3}{ds_1} ds_1$, $n_3 + \frac{dn_3}{ds_1} ds_1$, must be coplanar.

Accordingly the condition in question is

$$\begin{vmatrix} l_1, & m_1, & n_1 \\ l_3, & m_3, & n_3 \\ \frac{dl_3}{ds_1}, & \frac{dm_3}{ds_1}, & \frac{dn_3}{ds_1} \end{vmatrix} = 0. \qquad (18)$$

Again, the constituents of the above determinant must be connected by equations of the form

$$\frac{dl_3}{ds_1} + \alpha l_1 + \beta l_3 = 0,$$

$$\frac{dm_3}{ds_1} + \alpha m_1 + \beta m_3 = 0,$$

$$\frac{dn_3}{ds_1} + \alpha n_1 + \beta n_3 = 0.$$

If we multiply these equations by l_3, m_3, n_3, and add, we see that $\beta = 0$; and hence we get

$$\frac{dl_3}{ds_1} : \frac{dm_3}{ds_1} : \frac{dn_3}{ds_1} = l_1 : m_1 : n_1.$$

(Compare § 7.)

Again, if the curves s_1 and s_2 cut orthogonally we have

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0;$$

hence
$$l_2 \frac{dl_3}{ds_1} + m_2 \frac{dm_3}{ds_1} + n_2 \frac{dn_3}{ds_1} = 0,$$

or
$$l_3 \frac{dl_2}{ds_1} + m_3 \frac{dm_2}{ds_1} + n_3 \frac{dn_2}{ds_1} = 0. \tag{19}$$

Equation (19) may be written in the form

$$\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ \frac{dl_2}{ds_1}, & \frac{dm_2}{ds_1}, & \frac{dn_2}{ds_1}, \end{vmatrix} = 0. \tag{20}$$

This equation combined with the relation

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

expresses the condition that the equations $\alpha_1 = \text{const.}$ and $\alpha_2 = \text{const.}$ represent lines of curvature.

15. We can immediately prove the well-known theorem that if two surfaces, S and S', intersect at a constant angle, ϕ, and if their line of intersection (s_1) be a line of curvature on one, S, it is also a line of curvature on the other, S'. For, let l_3, m_3, n_3 be the direction cosines of the normal to the surface S, and l'_3, m'_3, n'_3 those of the normal to S', at any common point P, then we have

$$l_3 l_1 + m_3 m_1 + n_3 n_1 = 0, \tag{21}$$

$$l'_3 l_1 + m'_3 m_1 + n'_3 n_1 = 0,$$

$$l_3 l'_3 + m_3 m'_3 + n_3 n'_3 = \cos \phi = \text{const.} \tag{22}$$

Also
$$\frac{dl_3}{ds_1} : \frac{dm_3}{ds_1} : \frac{dn_3}{ds_1} = l_1 : m_1 : n_1. \tag{23}$$

Next, if we differentiate (22) we get, by aid of (23),

$$l_3 \frac{dl'_3}{ds_1} + m_3 \frac{dm'_3}{ds_1} + n_3 \frac{dn'_3}{ds_1} = 0.$$

We also have

$$l'_3 \frac{dl'_3}{ds_1} + m'_3 \frac{dm'_3}{ds_1} + n'_3 \frac{dn'_3}{ds_1} = 0.$$

Hence $$\frac{dl'_3}{ds_1} : \frac{dm'_3}{ds_1} : \frac{dn'_3}{ds_1} = n_3 m'_3 - n'_3 m_3 : l_3 n'_3 - l'_3 n_3 : m_3 l'_3 - m'_3 l_3.$$

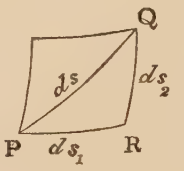
But, by (21), we also have

$$l_1 : m_1 : n_1 = n_3 m'_3 - n'_3 m_3 : l_3 n'_3 - l'_3 n_3 : m_3 l'_3 - m'_3 l_3,$$

Consequently $$\frac{dl'_3}{ds_1} : \frac{dm'_3}{ds_1} : \frac{dn'_3}{ds_1} = l_1 : m_1 : n_1,$$

and accordingly s_1 is a line of curvature on S'.

16. **Element of any Arc.**—If ds be the arc joining P and Q, two indefinitely near points, we have



$$ds^2 = ds_1^2 + ds_2^2 + 2ds_1 ds_2 \cos\omega$$
$$= k_1^2 da_1^2 + k_2^2 da_2^2 + 2k_1 k_2 \cos\omega \, da_1 da_2. \qquad (24)$$

17. **Radius of Curvature.**—We can now express the radius of curvature of any normal section of the surface.

For, let xyz be the coordinates of the point P, then the equation of the tangent plane at P may be written in the form

$$(X - x) l_3 + (Y - y) m_3 + (Z - z) n_3 = 0,$$

where X, Y, Z are the coordinates of any point on the plane; hence, if $x + \xi$, $y + \eta$, $z + \zeta$ be the coordinates of a point Q, we have

$$p = l_3 \xi + m_3 \eta + n_3 \zeta,$$

where p is the distance of Q from the tangent plane. Again, if Q is a point on the surface, indefinitely near to P, we may write

$$\xi = dx + \tfrac{1}{2} d^2x, \quad \eta = dy + \tfrac{1}{2} d^2y, \quad \zeta = dz + \tfrac{1}{2} d^2z,$$

and consequently

$$p = \tfrac{1}{2} (l_3 d^2x + m_3 d^2y + n_3 d^2z). \qquad (25)$$

As before, let $PQ = ds$, and let ρ be the radius of curvature of the normal section drawn through PQ; then we have

$$\frac{1}{\rho} = \frac{2p}{ds^2} = \frac{l_3 d^2x + m_3 d^2y + n_3 d^2z}{ds_1^2 + 2\cos\omega\, ds_1 ds_2 + ds_2^2}.$$

Again we have

$$dx = \frac{dx}{da_1} da_1 + \frac{dx}{da_2} da_2 = k_1 l_1 da_1 + k_2 l_2 da_2;$$

$$\therefore\ d^2x = \frac{d}{da_1}(k_1 l_1)\, da_1^2 + 2\frac{d(k_1 l_1)}{da_2} da_1 da_2 + \frac{d(k_2 l_2)}{da_2} da_2^2,$$

with similar values for d^2y and d^2z: consequently

$$l_3 d^2x + m_3 d^2y + n_3 d^2z = L_1 ds_1^2 + 2M_1 ds_1 ds_2 + 2N_1 ds_2^2, \qquad (26)$$

where

$$\left.\begin{aligned} L_1 &= l_3 \frac{dl_1}{ds_1} + m_3 \frac{dm_1}{ds_1} + n_3 \frac{dn_1}{ds_1} \\ M_1 &= l_3 \frac{dl_1}{ds_2} + m_3 \frac{dm_1}{ds_2} + n_3 \frac{dn_1}{ds_2} \\ N_1 &= l_3 \frac{dl_2}{ds_2} + m_3 \frac{dm_2}{ds_2} + n_3 \frac{dn_2}{ds_2} \end{aligned}\right\}. \qquad (27)$$

Hence we get

$$\frac{1}{\rho} = \frac{L_1 ds_1^2 + 2M_1 ds_1 ds_2 + N_1 ds_2^2}{ds_1^2 + 2\cos\omega\, ds_1 ds_2 + ds_2^2}. \qquad (28)$$

From this it follows, as already shown in § 11, that L_1 is the reciprocal of the radius of curvature of the normal section passing through ds_1, and N_1 the corresponding quantity for ds_2.

18. **Inflexional Tangents.**—Again, the directions of the inflexional tangents to the surface at P are given by the equation

$$L_1 ds_1^2 + 2M_1 ds_1 ds_2 + N_1 ds_2^2 = 0. \qquad (29)$$

Accordingly, the surface is *anticlastic* at P if M_1^2 is greater than $L_1 N_1$. The surface is developable if

$$L_1 N_1 = M_1^2$$

at all points.

19. If $M_1 = 0$ and $\omega = \frac{\pi}{2}$, the curves s_1 and s_2 are lines of curvature; in this case if R_1 and R_2 be the principal radii of curvature, we have

$$L_1 = \frac{1}{R_1}, \quad N_1 = \frac{1}{R_2}, \tag{30}$$

and equation (28) becomes

$$\frac{1}{\rho} = \frac{\cos^2\theta}{R_1} + \frac{\sin^2\theta}{R_2}, \tag{31}$$

where θ is the angle between ds and ds_1. This agrees with Euler's well-known equation.

20. **Indicatrix, Conjugate Curves.**—If P be taken as the origin, and the tangents to ds_1 and ds_2 as the axes of x and y, the equation of the *indicatrix* may, in general, be written

$$L_1 x^2 + 2M_1 xy + N_1 y^2 = \text{const.}$$

Consequently, the tangents to ds_1 and ds_2 are *conjugate tangents* whenever $M_1 = 0$. Hence the condition that the curves s_1 and s_2 should be conjugate is expressed by the equation

$$l_3 \frac{dl_1}{ds_2} + m_3 \frac{dm_1}{ds_2} + n_3 \frac{dn_1}{ds_2} = 0. \tag{32}$$

21. **Application to Ellipsoid.**—In the case of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

we have

$$l_3 = \frac{px}{a^2}, \quad m_3 = \frac{py}{b^2}, \quad n_3 = \frac{pz}{c^2}.$$

where p is the length of the perpendicular drawn from the centre to the tangent plane at the point xyz.

Hence, if ρ_1 be the radius of curvature of the normal section drawn through ds_1, we have

$$-\frac{1}{\rho_1} = l_1 \frac{dl_3}{ds_1} + m_1 \frac{dm_3}{ds_1} + n_1 \frac{dn_3}{ds_1}$$

$$= \frac{l_1}{a^2}\frac{d}{ds_1}(px) + \frac{m_1}{b^2}\frac{d}{ds_1}(py) + \frac{n_1}{c^2}\frac{d}{ds_1}(pz) = p\left(\frac{l_1^2}{a^2} + \frac{m_1^2}{b^2} + \frac{n_1^2}{c^2}\right),$$

since $$\frac{l_1 x}{a^2} + \frac{m_1 y}{b^2} + \frac{n_1 z}{c^2} = 0.$$

Hence, if D be the semidiameter drawn parallel to the tangent (l_1, m_1, n_1), we have

$$\rho_1 = -\frac{D^2}{p},$$

the well-known expression for the radius of curvature of any normal section to an ellipsoid (Salmon, *Geometry of Three Dimensions*, p. 168).

22. Next, let s_1 be a geodesic on the ellipsoid, then equations (10), Art. 4, give

$$\frac{dl_1}{ds_1} = \frac{l_3}{\rho_1} = -\frac{p^2}{D^2}\frac{x}{a^2}, \quad \frac{dm_1}{ds_1} = -\frac{p^2}{D^2}\frac{y}{b^2}, \quad \frac{dn_1}{ds_1} = -\frac{p^2}{D^2}\frac{z}{c^2}. \tag{33}$$

Accordingly,

$$\frac{l_1}{a^2}\frac{dl_1}{ds_1} + \frac{m_1}{b^2}\frac{dm_1}{ds_1} + \frac{n_1}{c^2}\frac{dn_1}{ds_1} = -\frac{p^2}{D^2}\left(\frac{x}{a^4}\frac{dx}{ds_1} + \frac{y}{b^4}\frac{dy}{ds_1} + \frac{z}{c^4}\frac{dz}{ds_1}\right) = -\tfrac{1}{2}\frac{p^2}{D^2}\frac{d}{ds_1}\left(\frac{1}{p^2}\right),$$

or $$\frac{d}{ds_1}\left(\frac{1}{D^2}\right) + \frac{p^2}{D^2}\frac{d}{ds_1}\left(\frac{1}{p^2}\right) = 0.$$

Hence we get at once Joachimsthal's well-known equation for a geodesic, viz.

$$pD = \text{const.} \tag{34}$$

If the geodesic passes through an umbilic, we have

$$pD = ac.$$

Again, if α_1 and α_2 be lines of curvature on the ellipsoid, we have

$$l_1 : m_1 : n_1 = \frac{dl_2}{ds_1} : \frac{dm_2}{ds_1} : \frac{dn_2}{ds_1}.$$

Hence, since

$$\frac{l_1 l_2}{a^2} + \frac{m_1 m_2}{b^2} + \frac{n_1 n_2}{c^2} = 0,$$

we get
$$\frac{l_2}{a^2}\frac{dl_2}{ds_1}+\frac{m_2}{b^2}\frac{dm_2}{ds_1}+\frac{n_2}{c^2}\frac{dn_2}{ds_1}=0. \tag{35}$$

Accordingly, D is constant along a line of curvature, where D is the semi-diameter drawn parallel to ds_2.

23. Again, the condition for conjugate tangents may be written
$$l_1\frac{dl_3}{ds_2}+m_1\frac{dm_3}{ds_2}+n_1\frac{dn_3}{ds_2}=0.$$

For the ellipsoid this becomes
$$\frac{l_1}{a^2}\frac{dx}{ds_2}+\frac{m_1}{b^2}\frac{dy}{ds_2}+\frac{n_1}{c^2}\frac{dz}{ds_2}=0,$$

or
$$\frac{l_1 l_2}{a^2}+\frac{m_1 m_2}{b^2}+\frac{n_1 n_2}{c^2}=0. \tag{36}$$

Hence the lines of curvature of an ellipsoid are determined by (36) along with
$$l_1 l_2+m_1 m_2+n_1 n_2=0.$$

Also, since
$$l_2 l_3+m_2 m_3+n_2 n_3=0,$$

the lines of curvature are determined by the equation
$$\begin{vmatrix} l_3, & m_3, & n_3 \\ l_1, & m_1, & n_1 \\ \frac{l_1}{a^2}, & \frac{m_1}{b^2}, & \frac{n_1}{c^2} \end{vmatrix}=0,$$

or by
$$\begin{vmatrix} x, & y, & z \\ \frac{dx}{ds}, & \frac{dy}{ds}, & \frac{dz}{ds} \\ a^2\frac{dx}{ds}, & b^2\frac{dy}{ds}, & c^2\frac{dz}{ds} \end{vmatrix}=0. \tag{37}$$

24. **Lines of Curvature.**—By the elementary theory of maxima and minima (*Differential Calculus*, Art. 144) we see immediately from (28) that the directions of the lines of curvature are given by the equation

$$ds_1^2(L_1 \cos\omega - M_1) + ds_1 ds_2(L_1 - N_1) + ds_2^2(M_1 - N_1 \cos\omega) = 0. \quad (38)$$

Also that the principal radii of curvature are the roots of the quadratic

$$\rho^2(L_1N_1 - M_1^2) - \rho(L_1 + N_1 - 2M_1\cos\omega) + \sin^2\omega = 0. \quad (39)$$

Hence, if, as before, R_1 and R_2 be the principal radii of curvature, we have

$$\frac{1}{R_1R_2} = \frac{L_1N_1 - M_1^2}{\sin^2\omega}. \quad (40)$$

25. **Measure of Curvature.**—Next, denoting the *measure of curvature* by M, we have

$$M = \frac{1}{R_1R_2} = \frac{L_1N_1 - M_1^2}{\sin^2\omega}. \quad (41)$$

This expression for the measure of curvature can be readily exhibited in a simple form. For we may write the quantities L_1, M_1, N_1 in the following determinant forms

$$L_1 = \frac{1}{k_1\sin\omega}\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ \frac{dl_1}{da_1}, & \frac{dm_1}{da_1}, & \frac{dn_1}{da_1} \end{vmatrix}, \quad N_1 = \frac{1}{k_2\sin\omega}\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ \frac{dl_2}{da_2}, & \frac{dm_2}{da_2}, & \frac{dn_2}{da_2} \end{vmatrix}. \quad (42)$$

$$M_1 = \frac{1}{k_2\sin\omega}\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ \frac{dl_1}{da_2}, & \frac{dm_1}{da_2}, & \frac{dn_1}{da_2} \end{vmatrix} = \frac{1}{k_1\sin\omega}\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ \frac{dl_2}{da_1}, & \frac{dm_2}{da_1}, & \frac{dn_2}{da_1} \end{vmatrix}. \quad (43)$$

The value of the measure of curvature can now be found by the multiplication of these determinants.

26. **Case of Orthogonal Curves.**—I shall first consider the case where α_1 and α_2 cut orthogonally. In this case we have $\omega = \frac{\pi}{2}$; hence

$$L_1 N_1 = \frac{1}{k_1 k_2} \begin{vmatrix} 1, & 0, & 0 \\ 0, & 1, & l_2 \frac{dl_1}{d\alpha_1} + m_2 \frac{dm_1}{d\alpha_1} + n_2 \frac{dn_1}{d\alpha_1} \\ l_1 \frac{dl_2}{d\alpha_2} + m_1 \frac{dm_2}{d\alpha_2} + n_1 \frac{dn_2}{d\alpha_2}, & 0, & \frac{dl_1}{d\alpha_1}\frac{dl_2}{d\alpha_2} + \frac{dm_1}{d\alpha_1}\frac{dm_2}{d\alpha_2} + \frac{dn_1}{d\alpha_1}\frac{dn_2}{d\alpha_2} \end{vmatrix}$$

$$= \frac{1}{k_1 k_2}\left(\frac{dl_1}{d\alpha_1}\frac{dl_2}{d\alpha_2} + \frac{dm_1}{d\alpha_1}\frac{dm_2}{d\alpha_2} + \frac{dn_1}{d\alpha_1}\frac{dn_2}{d\alpha_2}\right). \tag{44}$$

In like manner

$$M_1^2 = \frac{1}{k_1 k_2}\left(\frac{dl_1}{d\alpha_2}\frac{dl_2}{d\alpha_1} + \frac{dm_1}{d\alpha_2}\frac{dm_2}{d\alpha_1} + \frac{dn_1}{d\alpha_2}\frac{dn_2}{d\alpha_1}\right). \tag{45}$$

Hence, if α_1 and α_2 be lines of curvature, we have

$$\frac{dl_1}{d\alpha_2}\frac{dl_2}{d\alpha_1} + \frac{dm_1}{d\alpha_2}\frac{dm_2}{d\alpha_1} + \frac{dn_1}{d\alpha_2}\frac{dn_2}{d\alpha_4} = 0. \tag{46}$$

In general, if r_1 and r_2 be the geodesic radii of curvature, we have by (7),

$$\left.\begin{aligned} l_2 \frac{dl_1}{d\alpha_1} + m_2 \frac{dm_1}{d\alpha_1} + n_2 \frac{dn_1}{d\alpha_1} &= -\frac{k_1}{r_1} \\ \text{and} \qquad l_2 \frac{dl_1}{d\alpha_2} + m_2 \frac{dm_1}{d\alpha_2} + n_2 \frac{dn_1}{d\alpha_2} &= \frac{k_2}{r_2} \end{aligned}\right\}. \tag{47}$$

If we differentiate the former with respect to α_2, and the latter with respect to α_1, we get, by subtraction,

$$\frac{dl_1}{d\alpha_1}\frac{dl_2}{d\alpha_2} + \frac{dm_1}{d\alpha_1}\frac{dm_2}{d\alpha_2} + \frac{dn_1}{d\alpha_1}\frac{dn_2}{d\alpha_2} - \frac{dl_1}{d\alpha_2}\frac{dl_2}{d\alpha_1} - \frac{dm_1}{d\alpha_2}\frac{dm_2}{d\alpha_1} - \frac{dn_1}{d\alpha_2}\frac{dn_2}{d\alpha_1}$$

$$= -\frac{d}{d\alpha_2}\left(\frac{k_1}{r_1}\right) - \frac{d}{d\alpha_1}\left(\frac{k_2}{r_2}\right); \tag{48}$$

hence we get

$$\frac{1}{R_1 R_2} = -\frac{1}{k_1 k_2}\left\{\frac{d}{da_2}\left(\frac{k_1}{r_1}\right) + \frac{d}{da_1}\left(\frac{k_2}{r_2}\right)\right\}. \tag{49}$$

This agrees with the form given in Bertrand, *Cal. Diff.*, § 741.

The latter equation can be readily transformed by (15) into

$$\frac{d}{ds_2}\left(\frac{1}{r_1}\right) + \frac{d}{ds_1}\left(\frac{1}{r_2}\right) + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{R_1 R_2} = 0.$$

On the surface of a sphere this becomes

$$\frac{d}{ds_2}\left(\frac{1}{r_1}\right) + \frac{d}{ds_1}\left(\frac{1}{r_2}\right) + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{a^2} = 0, \tag{50}$$

where a is the radius of sphere.—Compare Bertrand, *Cal. Diff.*, § 558.

27. **General Expression for Measure of Curvature.**—In general we have, from (42),

$$L_1 N_1 = \frac{1}{k_1 k_2 \sin^2\omega}\begin{vmatrix} 1, & \cos\omega, & 0 \\ \cos\omega, & 1, & l_2\frac{dl_1}{da_1} + m_2\frac{dm_1}{da_1} + n_2\frac{dn_1}{da_1} \\ l_1\frac{dl_2}{da_2} + m_1\frac{dm_2}{da_2} + n_1\frac{dn_2}{da_2}, & 0, & \frac{dl_1}{da_1}\frac{dl_2}{da_2} + \frac{dm_1}{da_1}\frac{dm_2}{da_2} + \frac{dn_1}{da_1}\frac{dn_2}{da_2} \end{vmatrix} \tag{51}$$

and, from (43),

$$M_1^2 = \frac{1}{k_1 k_2 \sin^2\omega}\begin{vmatrix} 1, & \cos\omega, & 0 \\ \cos\omega, & 1, & l_2\frac{dl_1}{da_2} + m_2\frac{dm_1}{da_2} + n_2\frac{dn_1}{da_2} \\ l_1\frac{dl_2}{da_1} + m_1\frac{dm_2}{da_1} + n_1\frac{dn_2}{da_1}, & 0, & \frac{dl_1}{da_2}\frac{dl_2}{da_1} + \frac{dm_1}{da_2}\frac{dm_2}{da_1} + \frac{dn_1}{da_2}\frac{dn_2}{da_1} \end{vmatrix} \tag{52}$$

$$\text{Next, if } A_1 = l_1\frac{dl_2}{da_1} + m_1\frac{dm_2}{da_1} + n_1\frac{dn_2}{da_1}, \quad A_2 = l_1\frac{dl_2}{da_2} + m_1\frac{dm_2}{da_2} + n_1\frac{dn_2}{da_2}, \quad (53)$$

we may write equations (51) and (52) as follows :—

$$L_1 N_1 k_1 k_2 \sin^2\omega = \begin{vmatrix} 1, & \cos\omega, & 0 \\ \cos\omega, & 1, & -A_1 - \sin\omega\,\dfrac{d\omega}{da_1} \\ A_2, & 0, & \dfrac{dl_1}{da_1}\dfrac{dl_2}{da_2} + \dfrac{dm_1}{da_1}\dfrac{dm_2}{da_2} + \dfrac{dn_1}{da_1}\dfrac{dn_2}{da_2} \end{vmatrix}$$

$$= \sin^2\omega\left(\frac{dl_1}{da_1}\frac{dl_2}{da_2} + \frac{dm_1}{da_1}\frac{dm_2}{da_2} + \frac{dn_1}{da_1}\frac{dn_2}{da_2}\right) - A_2\cos\omega\left(A_1 + \sin\omega\,\frac{d\omega}{da_1}\right),$$

$$M_1^2 k_1 k_2 \sin^2\omega = \begin{vmatrix} 1, & \cos\omega, & 0 \\ \cos\omega, & 1, & -A_2 - \sin\omega\,\dfrac{d\omega}{da_2} \\ A_1, & 0, & \dfrac{dl_1}{da_2}\dfrac{dl_2}{da_1} + \dfrac{dm_1}{da_2}\dfrac{dm_2}{da_1} + \dfrac{dn_1}{da_2}\dfrac{dn_2}{da_1} \end{vmatrix}$$

$$= \sin^2\omega\left(\frac{dl_1}{da_2}\frac{dl_2}{da_1} + \frac{dm_1}{da_2}\frac{dm_2}{da_1} + \frac{dn_1}{da_2}\frac{dn_2}{da_1}\right) - A_1\cos\omega\left(A_2 + \sin\omega\,\frac{d\omega}{da_2}\right).$$

Again, we get from (53), by differentiation,

$$\frac{dA_2}{da_1} - \frac{dA_1}{da_2} = \frac{dl_1}{da_1}\frac{dl_2}{da_2} + \frac{dm_1}{da_1}\frac{dm_2}{da_2} + \frac{dn_1}{da_1}\frac{dn_2}{da_2} - \frac{dl_1}{da_2}\frac{dl_2}{da_1} - \frac{dm_1}{da_2}\frac{dm_2}{da_1} - \frac{dn_1}{da_2}\frac{dn_2}{da_1}.$$

Hence, we have

$$(L_1 N_1 - M_1^2)\sin^2\omega\; k_1 k_2 = \cos\omega\,\sin\omega\left(A_1\frac{d\omega}{da_2} - A_2\frac{d\omega}{da_1}\right) + \sin^2\omega\left(\frac{dA_2}{da_1} - \frac{dA_1}{da_2}\right),$$

$$\therefore M = \frac{1}{k_1 k_2 \sin\omega}\left\{\frac{1}{\sin\omega}\left(\frac{dA_2}{da_1} - \frac{dA_1}{da_2}\right) + \frac{1}{\sin^2\omega}\left(A_1\frac{d(\sin\omega)}{da_2} - A_2\frac{d(\sin\omega)}{da_1}\right)\right\}$$

$$= \frac{1}{k_1 k_2 \sin\omega}\left\{\frac{d}{da_1}\left(\frac{A_2}{\sin\omega}\right) - \frac{d}{da_2}\left(\frac{A_1}{\sin\omega}\right)\right\}. \quad (54)$$

Again, from (7) and (8) we have

$$A_2 = -\sin\omega\,\frac{k_2}{r_2}, \quad A_1 = \sin\omega\left(\frac{k_1}{r_1} - \frac{d\omega}{da_1}\right).$$

Hence we get, finally,

$$M = -\frac{1}{k_1 k_2 \sin\omega}\left\{\frac{d}{da_1}\left(\frac{k_2}{r_2}\right) + \frac{d}{da_2}\left(\frac{k_1}{r_1}\right) - \frac{d^2\omega}{da_1\,da_2}\right\}. \qquad (55)$$

This is equivalent to Gauss' well-known expression for the measure of curvature at any point on a surface (Liouvillé's *Monge*, p. 524; Salmon's *Geometry of Three Dimensions*, p. 384).

Again, as k_1, k_2, r_1, r_2 and ω are unaltered if the surface be supposed flexible, but not extensible, the measure of curvature is unaltered by such deformation.

The condition for a developable surface is represented by the equation

$$\frac{d}{da_1}\left(\frac{k_2}{r_2}\right) + \frac{d}{da_2}\left(\frac{k_1}{r_1}\right) - \frac{d^2\omega}{da_1\,da_2} = 0. \qquad (56)$$

Also, the *Curvatura integra* of any portion of a surface is represented by

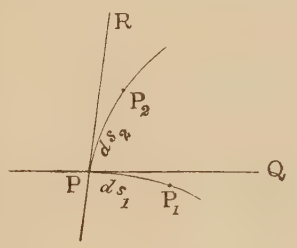
$$\iint \frac{k_1 k_2 \sin\omega\, da_1 da_2}{R_1 R_2} = -\iint\left\{\frac{d}{da_1}\left(\frac{k_2}{r_2}\right) + \frac{d}{da_1}\left(\frac{k_1}{r_1}\right) - \frac{d^2\omega}{da_1 da_2}\right\} da_1\, da_2$$

taken through that portion of the surface.

28. **Orthogonal Plane Curves.**—Applying our method to plane curves we suppose that

$$\phi_1(x, y) = a_1, \quad \phi_2(x, y) = a_2$$

represent two systems of orthogonal curves. Then, as before, ds_1 is an element of the curve $a_2 = \text{const.}$; now if ψ be the angle that the tangent makes with the axis of x, we have



$$\frac{dx}{da_1} = \frac{dx}{ds_1}\frac{ds_1}{da_1} = k_1\cos\psi, \quad \frac{dy}{da_1} = k_1\sin\psi.$$

Also, since the curves cut orthogonally, we have

$$\frac{dx}{da_2} = -k_2\sin\psi, \quad \frac{dy}{da_2} = k_2\cos\psi.$$

Again, if ρ_1 be the radius of curvature of α_1 at P, and ρ_2 that of α_2, we have

$$\frac{1}{\rho_1} = \frac{d\psi}{ds_2} = \frac{1}{k_2}\frac{d\psi}{d\alpha_2}, \quad \frac{1}{\rho_2} = -\frac{d\psi}{ds_1} = -\frac{1}{k_1}\frac{d\psi}{d\alpha_1}. \tag{57}$$

Also, since

$$\frac{d^2x}{d\alpha_1\, d\alpha_2} = \frac{d^2x}{d\alpha_2\, d\alpha_1},$$

we have

$$\frac{d}{d\alpha_2}(k_1 \cos\psi) + \frac{d}{d\alpha_1}(k_2 \sin\psi) = 0,$$

and also

$$\frac{d}{d\alpha_2}(k_1 \sin\psi) - \frac{d}{d\alpha_1}(k_2 \cos\psi) = 0,$$

or

$$\cos\psi\frac{dk_1}{d\alpha_2} - k_1 \sin\psi\frac{d\psi}{d\alpha_2} + \sin\psi\frac{dk_2}{d\alpha_1} + k_2\cos\psi\frac{d\psi}{d\alpha_1} = 0,$$

$$\sin\psi\frac{dk_1}{d\alpha_2} + k_1\cos\psi\frac{d\psi}{d\alpha_2} - \cos\psi\frac{dk_2}{d\alpha_1} + k_2\sin\psi\frac{d\psi}{d\alpha_1} = 0.$$

If we multiply the first of these by $\cos\psi$ and the second by $\sin\psi$ we get, by addition,

$$\frac{dk_1}{d\alpha_2} + k_2\frac{d\psi}{d\alpha_1} = 0, \text{ and likewise } \frac{dk_2}{d\alpha_1} = k_1\frac{d\psi}{d\alpha_2};$$

$$\therefore \quad \frac{dk_1}{d\alpha_2} = \frac{k_1 k_2}{\rho_2}, \quad \frac{dk_2}{d\alpha_1} = \frac{k_1 k_2}{\rho_1}. \tag{58}$$

Again, since

$$\frac{d^2\psi}{d\alpha_2\, d\alpha_1} = \frac{d^2\psi}{d\alpha_1\, d\alpha_2},$$

we have

$$\frac{d}{d\alpha_2}\left(\frac{1}{k_2}\frac{dk_1}{d\alpha_2}\right) + \frac{d}{d\alpha_1}\left(\frac{1}{k_1}\frac{dk_2}{d\alpha_1}\right) = 0. \tag{59}$$

This is, of course, a particular case of equation (49).

Also from (57) we have

$$\frac{d}{d\alpha_2}\left(\frac{k_1}{\rho_2}\right) + \frac{d}{d\alpha_1}\left(\frac{k_2}{\rho_1}\right) = 0.$$

Hence we get

$$\frac{1}{k_2}\frac{d}{da_2}\left(\frac{1}{\rho_2}\right)+\frac{1}{k_1}\frac{d}{da_1}\left(\frac{1}{\rho_1}\right)+\frac{1}{\rho_1^2}+\frac{1}{\rho_2^2}=0,$$

or

$$\frac{d}{ds_2}\left(\frac{1}{\rho_2}\right)+\frac{d}{ds_1}\left(\frac{1}{\rho_1}\right)+\frac{1}{\rho_1^2}+\frac{1}{\rho_2^2}=0. \qquad (60)$$

This remarkable theorem is also due to Lamé; it is, of course, a particular case of equation (50).

29. In the case where α_1 and α_2 are conjugate functions, we readily see that

$$\frac{dx}{da_1}=\frac{dy}{da_2}, \text{ and } \frac{dx}{da_2}=-\frac{dy}{da_1};$$

hence we get $k_1 = k_2$. Supposing either of these denoted by k, we have

$$\frac{1}{\rho_1}=-\frac{1}{k^2}\frac{dk}{da_1}=\frac{d\left(\frac{1}{k}\right)}{da_1}, \text{ and } \frac{1}{\rho_2}=\frac{d\left(\frac{1}{k}\right)}{da_2}.$$

Accordingly in this case

$$\frac{d}{da_2}\left(\frac{1}{\rho_1}\right)=\frac{d}{da_1}\left(\frac{1}{\rho_2}\right). \qquad (61)$$

SECTION 3.—*Application to Strain and Stress.*

30. I shall now proceed to apply the foregoing method to the determination of Lamé's general equations of strain and stress in curvilinear coordinates.

Adopting the usual notation, we shall represent by u, v, w the components of the displacement of the point xyz, taken, respectively, parallel to the axes of x, y, z.

Also let u_1, u_2, u_3 be the components of the displacement of xyz in the directions ds_1, ds_2, ds_3, respectively.

Then, we obviously have the following relations:—

$$\left.\begin{aligned} u &= l_1u_1 + l_2u_2 + l_3u_3 \\ v &= m_1u_1 + m_2u_2 + m_3u_3 \\ w &= n_1u_1 + n_2u_2 + n_3u_3 \end{aligned}\right\}, \qquad (1)$$

and also

$$\left.\begin{aligned} u_1 &= l_1 u + m_1 v + n_1 w \\ u_2 &= l_2 u + m_2 v + n_2 w \\ u_3 &= l_3 u + m_3 v + n_3 w \end{aligned}\right\}. \qquad (2)$$

Again, by the ordinary theory, the constituents of the small strain at P, relative to the rectangular system of axes OX, OY, OZ, are represented by the following expressions:—

$$\frac{du}{dx}, \frac{dv}{dy}, \frac{dw}{dz}, \tfrac{1}{2}\left(\frac{dw}{dy}+\frac{dv}{dz}\right), \tfrac{1}{2}\left(\frac{du}{dz}+\frac{dw}{dx}\right), \tfrac{1}{2}\left(\frac{dv}{dx}+\frac{du}{dy}\right).$$

We shall denote these respectively by the letters a, b, c, f, g, h.

Also let the six constituents of the strain relative to the orthogonal directions ds_1, ds_2, ds_3, be represented by a_1, b_1, c_1, f_1, g_1, h_1, respectively.

Let Δ be the cubical dilatation of the strain, and we have, as is well-known,

$$\Delta = \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}.$$

I shall commence by expressing Δ in terms of our curvilinear system of coordinates.

Here $$\Delta = \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz} = l_1\frac{du}{ds_1}+l_2\frac{du}{ds_2}+l_3\frac{du}{ds_3}$$

$$+ m_1\frac{dv}{ds_1}+m_2\frac{dv}{ds_2}+m_3\frac{dv}{ds_3}+n_1\frac{dw}{ds_1}+n_2\frac{dw}{ds_2}+n_3\frac{dw}{ds_3}.$$

Again, by (2) the latter expression is reduced to

$$\frac{du_1}{ds_1}+\frac{du_2}{ds_2}+\frac{du_3}{ds_3}-u\left(\frac{dl_1}{ds_1}+\frac{dl_2}{ds_2}+\frac{dl_3}{ds_3}\right)-v\left(\frac{dm_1}{ds_1}+\frac{dm_2}{ds_2}+\frac{dm_3}{ds_3}\right)-w\left(\frac{dn_1}{ds_1}+\frac{dn_2}{ds_2}+\frac{dn_3}{ds_3}\right).$$

If we now substitute for u, v, w, their values given in (1), we get immediately, by equations (19) and (21) of section 1,

$$\Delta = \frac{du_1}{ds_1} + \frac{du_2}{ds_2} + \frac{du_3}{ds_3} - u_1\left(\frac{1}{{}_2R_1} + \frac{1}{{}_3R_1}\right) - u_2\left(\frac{1}{{}_1R_2} + \frac{1}{{}_3R_2}\right) - u_3\left(\frac{1}{{}_1R_3} + \frac{1}{{}_2R_3}\right)$$

$$= \frac{1}{k_1k_2k_3}\left(\frac{d(u_1k_2k_3)}{da_1} + \frac{d(u_2k_1k_3)}{da_2} + \frac{d(u_3k_1k_2)}{da_3}\right). \qquad (3)$$

31. **Normal Strain.**—Again, by a well-known equation in strain, we have

$$a_1 = al_1^2 + bm_1^2 + cn_1^2 + 2fm_1n_1 + 2gn_1l_1 + 2hl_1m_1$$

$$= l_1\left(l_1\frac{du}{dx} + m_1\frac{du}{dy} + n_1\frac{du}{dz}\right) + m_1\left(l_1\frac{dv}{dx} + m_1\frac{dv}{dy} + n_1\frac{dv}{dz}\right)$$

$$+ n_1\left(l_1\frac{dw}{dx} + m_1\frac{dw}{dy} + n_1\frac{dw}{dz}\right)$$

$$= l_1\frac{du}{ds_1} + m_1\frac{dv}{ds_1} + n_1\frac{dw}{ds_1}$$

$$= \frac{du_1}{ds_1} - u\frac{dl_1}{ds_1} - v\frac{dm_1}{ds_1} - w\frac{dn_1}{ds_1}$$

$$= \frac{du_1}{ds_1} - (l_1u_1 + l_2u_2 + l_3u_3)\frac{dl_1}{ds_1} - (m_1u_1 + m_2u_2 + m_3u_3)\frac{dm_1}{ds_1}$$

$$- (n_1u_1 + n_2u_2 + n_3u_3)\frac{dn_1}{ds_1}$$

$$= \frac{du_1}{ds_1} - \frac{u_2}{{}_1R_2} - \frac{u_3}{{}_1R_3}.$$

In like manner,

$$b_1 = \frac{du_2}{ds_2} - \frac{u_1}{{}_2R_1} - \frac{u_3}{{}_2R_3},$$

$$c_1 = \frac{du_3}{ds_3} - \frac{u_1}{{}_3R_1} - \frac{u_2}{{}_3R_2}.$$

32. **Shearing Strain.**—Again, by the principle of the *elongation quadric* in the theory of strain, we have

$$\begin{aligned}
2h_1 &= l_1\left(l_2\frac{du}{dx}+m_2\frac{du}{dy}+n_2\frac{du}{dz}\right)+l_2\left(l_1\frac{du}{dx}+m_1\frac{du}{dy}+n_1\frac{du}{dz}\right)\\
&+m_1\left(l_2\frac{dv}{dx}+m_2\frac{dv}{dy}+n_2\frac{dv}{dz}\right)+m_2\left(l_1\frac{dv}{dx}+m_1\frac{dv}{dy}+n_1\frac{dv}{dz}\right)\\
&+n_1\left(l_2\frac{dw}{dx}+m_2\frac{dw}{dy}+n_2\frac{dw}{dz}\right)+n_2\left(l_1\frac{dw}{dx}+m_1\frac{dw}{dy}+n_1\frac{dw}{dz}\right)\\
&=l_1\frac{du}{ds_2}+l_2\frac{du}{ds_1}+m_1\frac{dv}{ds_2}+m_2\frac{dv}{ds_1}+n_1\frac{dw}{ds_2}+n_2\frac{dw}{ds_1}\\
&=\frac{du_1}{ds_2}+\frac{du_2}{ds_1}-u\left(\frac{dl_1}{ds_2}+\frac{dl_2}{ds_1}\right)-v\left(\frac{dm_1}{ds_2}+\frac{dm_2}{ds_1}\right)-w\left(\frac{dn_1}{ds_2}+\frac{dn_2}{ds_1}\right).
\end{aligned}$$

Substituting for u, v, w in terms of u_1, u_2, u_3, and proceeding as in the former case, we readily get

$$\begin{aligned}
2h_1 &= \frac{du_1}{ds_2}+\frac{du_2}{ds_1}+\frac{u_1}{{}_1R_2}+\frac{u_2}{{}_2R_1}\\
&=\frac{du_1}{ds_2}-\frac{u_1}{k_1}\frac{dk_1}{ds_2}+\frac{du_2}{ds_1}-\frac{u_2}{k_2}\frac{dk_2}{ds_1}\\
&=k_2\frac{d}{ds_1}\left(\frac{u_2}{k_2}\right)+k_1\frac{d}{ds_2}\left(\frac{u_1}{k_1}\right). \qquad (4)
\end{aligned}$$

In like manner

$$2g_1=k_1\frac{d}{ds_3}\left(\frac{u_1}{k_1}\right)+k_3\frac{d}{ds_1}\left(\frac{u_3}{k_3}\right),$$

$$2f_1=k_3\frac{d}{ds_2}\left(\frac{u_3}{k_3}\right)+k_2\frac{d}{ds_3}\left(\frac{u_2}{k_2}\right).$$

33. **Vortex Displacements.**—In the general theory of strain, if θ_1, θ_2, θ_3 be the small angular, or *vortex*, displacements at the point xyz, we have

$$2\theta_1=\frac{dw}{dy}-\frac{dv}{dz},\quad 2\theta_2=\frac{du}{dz}-\frac{dw}{dx},\quad 2\theta_3=\frac{dv}{dx}-\frac{du}{dy};$$

hence, if the vortex displacements relative to ds_1, ds_2, ds_3 be denoted respectively by Θ_1, Θ_2, Θ_3, we have

$$\Theta_1 = l_1\theta_1 + m_1\theta_2 + n_1\theta_3,$$
$$\Theta_2 = l_2\theta_1 + m_2\theta_2 + n_2\theta_3,$$
$$\Theta_3 = l_3\theta_1 + m_3\theta_2 + n_3\theta_3,$$

along with corresponding expressions for θ_1, θ_2, θ_3 in terms of Θ_1, Θ_2, Θ_3.

Accordingly,

$$2\Theta_1 = l_1\frac{dw}{dy} - m_1\frac{dw}{dx} + n_1\frac{dv}{dx} - l_1\frac{dv}{dz} + m_1\frac{du}{dz} - n_1\frac{du}{dy},$$

but

$$l_1\frac{dw}{dy} - m_1\frac{dw}{dx} = l_1\left(m_1\frac{dw}{ds_1} + m_2\frac{dw}{ds_2} + m_3\frac{dw}{ds_3}\right) - m_1\left(l_1\frac{dw}{ds_1} + l_2\frac{dw}{ds_2} + l_3\frac{dw}{ds_3}\right)$$
$$= n_3\frac{dw}{ds_2} - n_2\frac{dw}{ds_3}.$$

$$\therefore 2\Theta_1 = l_3\frac{du}{ds_2} + m_3\frac{dv}{ds_2} + n_3\frac{dw}{ds_2} - l_2\frac{du}{ds_3} - m_2\frac{dv}{ds_3} - n_2\frac{dw}{ds_3}$$
$$= \frac{du_3}{ds_2} - \frac{du_2}{ds_3} + u\left(\frac{dl_2}{ds_3} - \frac{dl_3}{ds_2}\right) + v\left(\frac{dm_2}{ds_3} - \frac{dm_3}{ds_2}\right) + w\left(\frac{dn_2}{ds_3} - \frac{dn_3}{ds_2}\right).$$

If we substitute by (1), we readily get

$$2\Theta_1 = \frac{du_3}{ds_2} - \frac{du_2}{ds_3} + \frac{u_2}{{}_2R_3} - \frac{u_3}{{}_3R_2}.$$

Again,

$$\frac{du_3}{ds_2} - \frac{u_3}{{}_3R_2} = \frac{du_3}{ds_2} + \frac{u_3}{k_3}\frac{dk_3}{ds_2} = \frac{1}{k_3}\frac{d(k_3u_3)}{ds_3},$$

hence

$$\left.\begin{aligned} 2\Theta_1 &= \frac{1}{k_3}\frac{d(u_3k_3)}{ds_2} - \frac{1}{k_2}\frac{d(u_2k_2)}{ds_3} \\ 2\Theta_2 &= \frac{1}{k_1}\frac{d(u_1k_1)}{ds_3} - \frac{1}{k_3}\frac{d(u_3k_3)}{ds_1} \\ 2\Theta_3 &= \frac{1}{k_2}\frac{d(u_2k_2)}{ds_1} - \frac{1}{k_1}\frac{d(u_1k_1)}{ds_2} \end{aligned}\right\}. \qquad (5)$$

From these we infer that the conditions in curvilinear coordinates for a pure strain are

$$\frac{d(u_3k_3)}{da_1}=\frac{d(u_1k_1)}{da_3},\quad \frac{d(u_3k_3)}{da_2}=\frac{d(u_2k_2)}{da_3},\quad \frac{d(u_1k_1)}{da_2}=\frac{d(u_2k_2)}{da_1}.$$

These latter conditions, however, can be established in a simpler manner from other considerations.

In polar coordinates the vortex displacements immediately become

$$\left.\begin{aligned} 2\Theta_1 &= \frac{1}{r\sin\theta}\left\{\frac{d(u_3\sin\theta)}{d\theta}-\frac{du_2}{d\phi}\right\} \\ 2\Theta_2 &= \frac{1}{r}\left\{\frac{1}{\sin\theta}\frac{du_1}{d\phi}-\frac{d(u_3r)}{dr}\right\} \\ 2\Theta_3 &= \frac{1}{r}\left(\frac{d(u_2r)}{dr}-\frac{du_1}{d\theta}\right) \end{aligned}\right\}.$$

34. **Application to Hydrodynamics.**—In hydrodynamics u, v, w denote the component velocities for xyz at any instant; hence, the equation of continuity, in polar coordinates, becomes

$$\frac{1}{\rho}\frac{d}{dt}\rho+\frac{1}{r^2}\left\{\frac{d(u_1r^2)}{dr}+\frac{r}{\sin\theta}\frac{d(u_2\sin\theta)}{d\theta}+\frac{r}{\sin\theta}\frac{du_3}{d\phi}\right\}=0.$$

Also, Θ_1, Θ_2, Θ_3, the components of the vortex rotation at the point are given by

$$2\Theta_1=\frac{1}{r\sin\theta}\left\{\frac{d(u_3\sin\theta)}{d\theta}-\frac{du_2}{d\phi}\right\},$$

$$2\Theta_2=\frac{1}{r}\left\{\frac{1}{\sin\theta}\frac{du_1}{d\phi}-\frac{d(ru_3)}{dr}\right\},$$

$$2\Theta_3=\frac{1}{r}\left\{\frac{d(u_2r)}{dr}-\frac{du_1}{d\theta}\right\}.$$

35. **Equations of Equilibrium for Isotropic Bodies.**—The general equations of equilibrium, in rectangular coordinates, for isotropic bodies may be written

$$(\kappa + \tfrac{4}{3}\mu)\frac{d\Delta}{dx} + 2\mu\left(\frac{d\theta_2}{dz} - \frac{d\theta_3}{dy}\right) + \rho X = 0,$$

$$(\kappa + \tfrac{4}{3}\mu)\frac{d\Delta}{dy} + 2\mu\left(\frac{d\theta_3}{dx} - \frac{d\theta_1}{dz}\right) + \rho Y = 0,$$

$$(\kappa + \tfrac{4}{3}\mu)\frac{d\Delta}{dz} + 2\mu\left(\frac{d\theta_1}{dy} - \frac{d\theta_2}{dx}\right) + \rho Z = 0.$$

Hence, multiplying by l_1, m_1, n_1, respectively, and adding we get

$$(\kappa + \tfrac{4}{3}\mu)\frac{d\Delta}{ds_1} + 2\mu\left\{l_1\left(\frac{d\theta_2}{dz} - \frac{d\theta_3}{dy}\right) + m_1\left(\frac{d\theta_3}{dx} - \frac{d\theta_1}{dz}\right) + n_1\left(\frac{d\theta_1}{dy} - \frac{d\theta_2}{dx}\right)\right\} + \rho\Sigma_1 = 0,$$

where Σ_1 is the component of the external force parallel to ds_1.

Again, as in § 33, we get immediately,

$$l_1\left(\frac{d\theta_2}{dz} - \frac{d\theta_3}{dy}\right) + m_1\left(\frac{d\theta_3}{dx} - \frac{d\theta_1}{dz}\right) + n_1\left(\frac{d\theta_1}{dy} - \frac{d\theta_2}{dx}\right)$$

$$= l_3\frac{d\theta_1}{ds_2} + m_3\frac{d\theta_2}{ds_2} + n_3\frac{d\theta_3}{ds_2} - l_2\frac{d\theta_1}{ds_3} - m_2\frac{d\theta_2}{ds_3} - n_2\frac{d\theta_3}{ds_3}$$

$$= \frac{d\Theta_3}{ds_2} - \frac{d\Theta_2}{ds_3} + \theta_1\left(\frac{dl_2}{ds_3} - \frac{dl_3}{ds_2}\right) + \theta_2\left(\frac{dm_2}{ds_3} - \frac{dm_3}{ds_2}\right) + \theta_3\left(\frac{dn_2}{ds_3} - \frac{dn_3}{ds_2}\right)$$

$$= \frac{1}{k_3}\frac{d(k_3\Theta_3)}{ds_2} - \frac{1}{k_2}\frac{d(k_2\Theta_2)}{ds_3}.$$

Hence the equations of equilibrium in curvilinear coordinates become, for an isotropic substance,

$$(\kappa + \tfrac{4}{3}\mu)\frac{d\Delta}{ds_1} + 2\mu\left\{\frac{1}{k_3}\frac{d(k_3\Theta_3)}{ds_2} - \frac{1}{k_2}\frac{d(k_2\Theta_2)}{ds_3}\right\} + \rho\Sigma_1 = 0,$$

$$(\kappa + \tfrac{4}{3}\mu)\frac{d\Delta}{ds_2} + 2\mu\left\{\frac{1}{k_1}\frac{d(k_1\Theta_1)}{ds_3} - \frac{1}{k_3}\frac{d(k_3\Theta_3)}{ds_1}\right\} + \rho\Sigma_2 = 0,$$

$$(\kappa + \tfrac{4}{3}\mu)\frac{d\Delta}{ds_3} + 2\mu\left\{\frac{1}{k_2}\frac{d(k_2\Theta_2)}{ds_1} - \frac{1}{k_1}\frac{d(k_1\Theta_1)}{ds_2}\right\} + \rho\Sigma_3 = 0.$$

These become, in polar coordinates,

$$(\kappa + \tfrac{4}{3}\mu)\frac{d\Delta}{dr} + \frac{2\mu}{r\sin\theta}\left\{\frac{d(\Theta_3 \sin\theta)}{d\theta} - \frac{d\Theta_2}{d\phi}\right\} + \rho R_1 = 0,$$

$$(\kappa + \tfrac{4}{3}\mu)\frac{1}{r}\frac{d\Delta}{d\theta} + \frac{2\mu}{r}\left\{\frac{d\Theta_1}{\sin\theta\, d\phi} - \frac{d(r\Theta_3)}{dr}\right\} + \rho R_2 = 0,$$

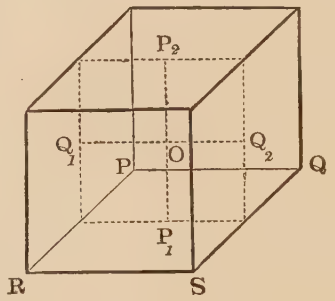
$$(\kappa + \tfrac{4}{3}\mu)\frac{d\Delta}{r\sin\theta d\phi} + \frac{2\mu}{r}\left\{\frac{d(r\Theta_2)}{dr} - \frac{d\Theta_1}{d\theta}\right\} + \rho R_3 = 0.$$

I shall conclude with a demonstration of the general equations of equilibrium for any elastic system.

36. **General Equations of Equilibrium in Rectangular Coordinates.**—We shall denote the six constituents of stress at any point for our rectangular system of coordinates, by

$$A, \quad B, \quad C, \quad F, \quad G, \quad H,$$

where A, B, C are the three normal stresses, and F, G, H the shearing stresses, at the point.



We commence by determining the general equations of equilibrium for an elementary parallepiped. In this case, let x, y, z be the coordinates of O, the centre of the parallelepiped; also let dx, dy, dz be the lengths of the edges of the parallelepiped; then, neglecting quantities which are indefinitely small in comparison with those which are retained, the system of stresses which act on the three elementary planes that pass through O, are

(1) $A\,dy\,dz$, $H\,dz\,dx$, $G\,dx\,dy$

parallel to the axis of x,

(2) $H\,dy\,dz$, $B\,dz\,dx$, $F\,dx\,dy$

parallel to the axis of y,

(3) $G\,dy\,dz$, $F\,dz\,dx$, $C\,dx\,dy$

parallel to the axis of z.

Again, the stresses on the several faces of the parallelepiped may be regarded as respectively acting at the middle points of its six faces.

Hence, the stresses which act at the middle points of the faces that are situated at the *positive* sides of the point O are,

$$(1)\quad \left(A+\tfrac{1}{2}\frac{dA}{dx}\,dx\right)dy\,dz,\quad \left(H+\tfrac{1}{2}\frac{dH}{dy}\,dy\right)dz\,dx,\quad \left(G+\tfrac{1}{2}\frac{dG}{dz}\,dz\right)dx\,dy,$$

parallel to the axis of x,

$$(2)\quad \left(H+\tfrac{1}{2}\frac{dH}{dx}\,dx\right)dy\,dz,\quad \left(B+\tfrac{1}{2}\frac{dB}{dy}\,dy\right)dz\,dx,\quad \left(F+\tfrac{1}{2}\frac{dF}{dz}\,dz\right)dx\,dy,$$

parallel to the axis of y, and

$$(3)\quad \left(G+\tfrac{1}{2}\frac{dG}{dx}\,dx\right)dy\,dz,\quad \left(F+\tfrac{1}{2}\frac{dF}{dy}\,dy\right)dz\,dx,\quad \left(C+\tfrac{1}{2}\frac{dC}{dz}\,dz\right)dx\,dy,$$

parallel to the axis of z.

Again the stresses which act on the opposite faces are

$$\left(A-\tfrac{1}{2}\frac{dA}{dx}\,dx\right)dy\,dz,\quad \left(H-\tfrac{1}{2}\frac{dH}{dy}\,dy\right)dx\,dz,\ \&c.$$

If Xdm, Ydm, Zdm be the components of the external force which acts on the element, we get at once the equations of equilibrium in the well-known form

$$\left.\begin{aligned}\frac{dA}{dx}+\frac{dH}{dy}+\frac{dG}{dz}+\rho X&=0\\ \frac{dH}{dx}+\frac{dB}{dy}+\frac{dF}{dz}+\rho Y&=0\\ \frac{dG}{dx}+\frac{dF}{dy}+\frac{dC}{dz}+\rho Z&=0\end{aligned}\right\},\qquad (6)$$

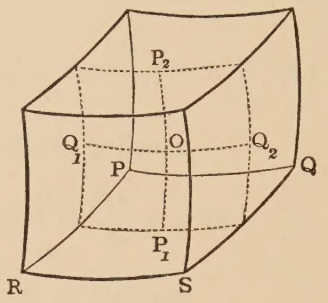
where ρ denotes the density at P.

37. **Equations in Curvilinear Coordinates.**—We proceed to give the corresponding investigation for the equilibrium of a curvilinear element whose edges are elements of lines of curvature belonging to our system of surfaces α_1, α_2, α_3.

We shall denote the six constituents of stress, at the central point O of our curvilinear element of volume, by

$$A_1,\ B_1,\ C_1,\quad F_1,\ G_1,\ H_1,$$

where each, as before, is referred to the unit of area. We shall also denote by $l_1,\ m_1,\ n_1$ the direction cosines of the tangent to OP_2 at O; and by $l_2,\ m_2,\ n_2,\ :\ l_3,\ m_3,\ n_3$, the corresponding direction cosines of the tangents to the two other curves of our system, which intersect at O. Then, the stresses at O are represented by



(1) $A_1k_2k_3da_2da_3,\ H_1k_1k_3da_1da_3,\ G_1k_1k_2da_1da_2$, parallel to the line l_1, m_1, n_1.

(2) $H_1k_2k_3da_2da_3,\ B_1k_1k_3da_1da_3,\ F_1k_1k_2da_1da_2$, parallel to $l_2,\ m_2,\ n_2$,

(3) $G_1k_2k_3da_2da_3,\ F_1k_1k_3da_1da_3,\ C_1k_1k_2da_1da_2$, parallel to $l_3,\ m_3,\ n_3$.

Hence the system of stresses which act on the faces on the *positive* sides of O are

(1) $$A_1k_2k_3da_2da_3 + \tfrac{1}{2}\frac{d(A_1k_2k_3)}{da_1}da_1da_2da_3,$$

$$H_1k_1k_3da_1da_3 + \tfrac{1}{2}\frac{d(H_1k_1k_3)}{da_2}da_1da_2da_3,$$

$$G_1k_1k_2da_1da_2 + \tfrac{1}{2}\frac{d(G_1k_1k_2)}{da_3}da_1da_2da_3,$$

corresponding to the first system;

(2) $$\left(H_1k_2k_3 + \tfrac{1}{2}\frac{d(H_1k_2k_3)}{da_1}da_1\right)da_2da_3,$$

$$\left(B_1k_1k_3 + \tfrac{1}{2}\frac{d(B_1k_1k_3)}{da_2}da_2\right)da_1da_3,$$

$$\left(F_1k_1k_2 + \tfrac{1}{2}\frac{d(F_1k_1k_2)}{da_3}da_3\right)da_1\,da_2,$$

corresponding to the second system;

$$(3)\quad \left(G_1k_2k_3 + \tfrac{1}{2}\frac{d(G_1k_2k_3)}{da_1}\,da_1\right)da_2da_3,$$

$$\left(F_1k_1k_3 + \tfrac{1}{2}\frac{d(F_1k_1k_3)}{da_2}\,da\right)da_1da_3,$$

$$\left(C_1k_1k_2 + \tfrac{1}{2}\frac{d(C_1k_1k_2)}{da_3}\,da_3\right)da_1da_2,$$

corresponding to the third system.

The systems of stresses acting respectively on the opposite faces of the curvilinear parallelepiped are obtained by changing the signs of the second terms in each of the preceding expressions from + to −.

We shall now proceed to find the component of all these stresses when taken in the direction represented by $l_1m_1n_1$. In finding this we shall have to multiply each stress component by the cosine of the angle which its line of action makes with the direction $l_1m_1n_1$. Before doing so we shall find it convenient to write down the following equations (already given):—

$$l_1\frac{dl_2}{ds_1} + m_1\frac{dm_2}{ds_1} + n_1\frac{dn_2}{ds_1} = -\frac{1}{{}_1R_2},$$

$$l_1\frac{dl_2}{ds_2} + m_1\frac{dm_2}{ds_2} + n_1\frac{dn_2}{ds_2} = \frac{1}{{}_2R_1},$$

$$l_1\frac{dl_3}{ds_2} + m_1\frac{dm_3}{ds_2} + n_1\frac{dn_3}{ds_2} = 0.$$

We shall, throughout, neglect all expressions higher than the *third order* in our small quantities. Hence, omitting the constant multiplier $da_1da_2da_3$, we see that the forces of the system (1) give

$$\frac{d}{da_1}(A_1k_2k_3),\quad \frac{d}{da_2}(H_1k_1k_3),\quad \frac{d}{da_3}(G_1k_1k_2).$$

Again, the system (2) gives

$$H_1k_2k_3\left(l_1\frac{dl_2}{ds_1} + m_1\frac{dm_2}{ds_1} + n_1\frac{dn_2}{ds_1}\right)k_1,\ \text{or}\ -H_1\frac{k_1k_2k_3}{{}_1R_2},$$

$$B_1k_1k_3\left(l_1\frac{dl_2}{ds_2} + m_1\frac{dm_2}{ds_2} + n_1\frac{dn_2}{ds_2}\right)k_2,\ \text{or}\quad B_1\frac{k_1k_2k_3}{{}_2R_1},$$

and $$F_1 k_1 k_2 \left(l_1 \frac{dl_2}{ds_3} + m_1 \frac{dm_2}{ds_3} + n_1 \frac{dn_2}{ds_3} \right) k_3, \text{ or zero.}$$

In like manner the system (3) gives

$$- G_1 \frac{k_1 k_2 k_3}{{}_1R_3}, \ 0, \text{ and } C_1 \frac{k_1 k_2 k_3}{{}_3R_1}.$$

Hence we get, finally,

$$\frac{1}{k_1 k_2 k_3} \left\{ \frac{d(A_1 k_2 k_3)}{da_1} + \frac{d(H_1 k_1 k_3)}{da_2} + \frac{d(G_1 k_1 k_2)}{da_3} \right\}$$
$$+ \frac{B_1}{{}_2R_1} + \frac{C_1}{{}_3R_1} - \frac{H_1}{{}_1R_2} - \frac{G_1}{{}_1R_3} + \rho \Xi_1 = 0,$$

along with the two similar equations,

$$\frac{1}{k_1 k_2 k_3} \left\{ \frac{d(H_1 k_2 k_3)}{da_1} + \frac{d(B_1 k_1 k_3)}{da_2} + \frac{d(F_1 k_1 k_2)}{da_3} \right\}$$
$$+ \frac{A_1}{{}_1R_2} + \frac{C_1}{{}_3R_2} - \frac{F_1}{{}_2R_3} - \frac{H_1}{{}_2R_1} + \rho \Xi_2 = 0,$$

$$\frac{1}{k_1 k_2 k_3} \left\{ \frac{d(G_1 k_2 k_3)}{da_1} + \frac{d(F_1 k_1 k_3)}{da_2} + \frac{d(C_1 k_1 k_2)}{da_3} \right\}$$
$$+ \frac{A_1}{{}_1R_3} + \frac{B_1}{{}_2R_3} - \frac{F_1}{{}_3R_2} - \frac{G_1}{{}_3R_1} + \rho \Xi_3 = 0.$$

These can be readily identified with the equations given by Lamé.

In polar coordinates the preceding become

$$\frac{1}{r^2} \frac{d(A_1 r^2)}{dr} + \frac{1}{r \sin \theta} \left(\frac{d(H_1 \sin \theta)}{d\theta} + \frac{dG_1}{d\phi} \right) - \frac{B_1 + C_1}{r} + \rho \Xi_1 = 0,$$

$$\frac{1}{r^2} \frac{d(H_1 r^2)}{dr} + \frac{1}{r \sin \theta} \left\{ \frac{d(B_1 \sin \theta)}{d\theta} + \frac{dF_1}{d\phi} \right\} + \frac{H_1}{r} - \frac{C_1}{r} \cot \theta + \rho \Xi_2 = 0,$$

$$\frac{1}{r^2} \frac{d(G_1 r^2)}{dr} + \frac{1}{r \sin \theta} \left\{ \frac{d(F_1 \sin \theta)}{d\theta} + \frac{dC_1}{d\phi} \right\} + \frac{G_1}{r} + \frac{F_1}{r} \cot \theta + \rho \Xi_3 = 0.$$